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C'est dans le but de connaître les concepts et opérations mathématiques des enfants que l'auteure a observé, sur une période de deux ans, la structure conceptuelle d'enfants et leur évolution. Des extraits de rencontres avec deux enfants de six ans sont analysés afin d'illustrer la construction du sens des nombres et des mots les représentant. Par ces expérimentations, l'auteure désire contribuer à l'élaboration d'un modèle des constructions cognitives des enfants en mathématiques. Les programmes éducatifs pourront alors être élaborés en fonction des mathématiques des enfants plutôt que de celles des adultes.

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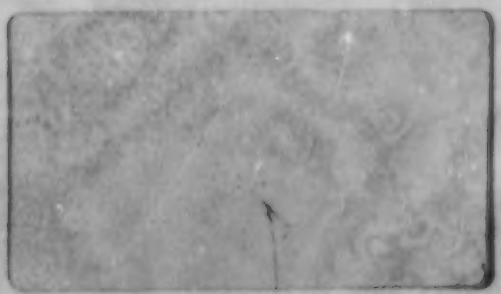

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A constructivist approach to the learning of arithmetic: Analysis of examples

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Séminaire sur la Représentation
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Children's Construction of Meaning for Arithmetical Words A Curriculum Problem[1]

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This paper was written for the Conference on Implicit and Explicit Knowledge in Science and Mathematics held at Tel Aviv University, Tel Aviv, Israel in October of 1988. The research on which the paper is based was supported by the National Science Foundation under Grant No. SED 80-16562, the Department of Mathematics Education, and the Institute for Behavioral Research of the University of Georgia. All opinions and findings are those of the author's and are not necessarily representative of the sponsoring agencies.

In the context of a curriculum improvement study conducted by the Center for the Learning and Teaching of Elementary Subjects at Michigan State University, I was asked to review a mathematics textbook series that is widely used in the United States. One of my purposes for doing the review was to search for indicators that the authors based their learning materials regarding number on a theory of how children construct the number sequence. The results of my search served as an inspiration for writing this paper and led me to recall an experience as a participant in developing a mathematics curriculum for the elementary school when I was at the University of Wisconsin. The interesting thing was that we mounted a concomitant research program where curriculum and theory development were to mutually overlap, each providing problems and insights for the other. It should be no surprise that our attempts to articulate the two activities fell far short of my expectations. This earlier experience led me to forego curriculum development and to concentrate on child studies in an attempt to formulate viable models of children's mathematical concepts and operations. At the time, I felt that such models were desperately needed in mathematics education because our curriculum development efforts were based on how we adults understood mathematics.

How we organized the curriculum for the numbers through twelve was almost the same as did the authors of the textbook series I reviewed in the curriculum improvement study. In the more recent textbook series, given a model set (a pictorial display), the children are to learn the relation "as many as" by drawing lines between the pictorial items and the items of several other (usually three) pictorial displays. In the case of "one" through "four", the children are expected to look at a pictorial display of perceptual items and learn to recognize the display by saying a number word and then, based on the relation "as many as", know that four is the number of any other set equivalent to the model set. To develop meaning for "five" through "twelve", the relation "one more than" is used. By imagining putting one more perceptual item with a pictorial display of five items, the children are to learn the meaning for "six", and etc. The "one more than" activities are correlated with counting perceptual items -- to find "how many" birds are in a picture of birds, the children are to count the birds -- and counting is the only difference in how we and the authors of the more recent textbook series organized the work on the numbers through twelve.

This approach to number is based on cardinal number theory, where a cardinal number is taken to be an object that is assigned to each set in such a way that equivalent sets, and equivalent sets only, have the same object corresponding to them (Hausdorff, 1962, p. 28). In my work with children, I have found an "ordinal approach" to number to be more appropriate for reasons that are given later in the paper. But this is not my essential point because to base the curriculum on either of these two approaches is

tantamount to basing the curriculum on how adults understand mathematics and on a view of the mathematics education of children as a process of transferring information, which is written in mathematics textbooks, to children without taking into account **children's mathematics**. The main thrust of my work, and I want it to be understood in this way, is to specify **numerical concepts and operations of children and make those the conceptual foundations of number in school mathematics**.

Children's Mathematics

Individuals who believe that mathematics is the way it is rather than the way human beings make it to be might reject my work out of hand because they may not believe that mathematics of children is a legitimate mathematics. I find this to be particularly unsettling, because in my way of thinking, the mathematical knowledge of the other must be taken as being relative to one's own frame of reference. It can be known through interpreting the language and actions of the other—by forming a possible conceptual model. These models are understood as springing from the conceptual operations that are available to the knower. They are nothing other than a constellation of available conceptual operations that are formed in organizing, or making sense of, experiential encounters with children. Such models constitute mathematics **of** children even though they are not taken to characterize how the mathematical knowledge of children really is. They are taken as a **fit** rather than a **match** (von Glasersfeld, 1983). So, to believe that the mathematics of children is not a legitimate mathematics is to essentially reject one's own mathematical concepts and operations.

Mathematics **for** a child consists of those mathematical concepts and operations that one hypothesizes the child might learn. These hypotheses might be formulated in such a way that they constitute a model something like what Vygotsky (1956) has called the child's *zone of potential development*—that which the child is capable of learning with the help of a teacher. But there are differences in my characterization of mathematics for children and Vygotsky's notion of a zone of potential development because mathematics for children is of the same basic nature as mathematics of children. I base mathematics for children on **experiential abstractions** from interactive communication with children. In the development of mathematics for children, at some point an adult makes hypotheses about what a child might learn based on past experience with the given child and without having tested how other children might alter a model of elements of the child's mathematical knowledge. But I would not call these hypotheses mathematics for children any more than I would consider my

knowledge of calculus to be mathematics for, say, a ten year-old. My choice is to think of mathematics **for** particular children to be mathematics **of** some other children.

Children's mathematics and solipsism

Because mathematics of and for children are concepts of the observer, it might seem that what I propose is nothing but solipsism--the theory that one's own knowledge is the only thing that can be known and verified. It has been interpreted in this way by at least two authors (Vergnaud, 1987; Wheeler, 1987). However, von Foerster (1984) has explicitly rejected solipsism.

> My arguments make, I understand, a most unpopular claim. One way of sweeping it under the rug is to dismiss it as just another attempt to rescue "solipsism," the view that this world is only in my imagination and the only reality is the imagining "I". Indeed, that was precisely what I was saying before, but I was talking only about a single organism. The situation is quite different when there are two, as I shall demonstrate... [T]he solipsistic claim falls to pieces when besides me I invent another autonomous organism (p. 59).

Children's mathematics **as I perceive it** is my invention. But, according to von Foerster's analysis, children's mathematics is not only in my imagination for if I agree, say, that there is another organism not unlike myself, I have to agree that the other organism can insist that their mathematical reality is the sole mathematical reality and that everything else is in **their** imagination, including my own mathematical reality. However, I know that my mathematical reality is not simply a concoction of the other's imagination and so I have to accept the other's mathematical reality as being distinct from my own. However, having no direct access to the other's reality, the only way I can know it is to make a model of it.

So, we must come to know the mathematics of and for children through intensive and extensive experiential encounters, and "to know" means to make models of their evolving mathematical concepts and operations. I emphasize "evolving" because children's mathematics is not static; it is to be viewed as the mathematics children construct in experiential contexts. What I do in this paper is to discuss some of my current understandings of children's numerical knowledge and indicate how it is strikingly different that what appears in school mathematics textbooks.

The models I construct of children's evolving mathematical concepts and operations are based in part on what I observe in the context of actually teaching children. These teaching experiments are exploratory and are aimed at discovering

what might go on in children's heads. The methodology of the teaching experiment (Steffe, 1983; Cobb & Steffe, 1983; Hunting, 1983; Thompson, 1982) is derived from Piaget's clinical interview, but because it also involves experimentation with the ways and means of influencing children's knowledge, it is more than a clinical interview. The clinical interview is aimed at establishing where children are, but the teaching experiment is directed toward understanding the progress children make over extended periods of time where there is no intention of teaching a predetermined way of operating.

In the following discussions of children's meanings for number words, I present specimens of interactive communication between myself (or other researchers working with me) and individual children along with interpretations of the specimens. The interpretations are not based solely on the particular specimens, but on a sequence of teaching episodes. I cannot emphasize this important point enough, because a specimen represents only a small segment of a one-half hour video tape of a teaching episode and over 60 of these video tapes exist for each child.

Brenda's Construction of Meanings for Number Words

Consider the following interchange with a six year old child, Brenda[2]. I presented her with a task where four of a collection of squares were covered by a cloth (three were visible) and told her that four of them were hidden and asked her to find how many there were in all. She first tried to raise the cloth to see the squares, but I asked her to try to find out without looking at the ones covered. She counted the three visible squares ("B" is used for "Brenda" and "T" is used for "teacher").

Protocol 1

B: 1-2-3 (touches each visible square in turn).

T: There are four here (taps the cloth).

B: (Lifts the cloth, revealing two squares) 4-5. (She touches each of these squares and puts the cloth back).

T: OK, I will show you two of them (folds back the cloth to reveal two of the four covered squares). There are four here, you count them.

B: 1-2-3-4-5 (touches each visible square in turn).

T: There are two more here (taps the cloth).

B: (Attempts to lift the cloth, so T removes it). 6-7 (touches the last two squares).

The meaning of "four" for Brenda is certainly at issue. Her attempt to lift the cloth indicates that she was aware of the hidden squares and intended to count the collection of squares. But her awareness of the hidden squares did not lead to counting.

Another interchange provides further insight into her possible meaning of number words. I covered six of a collection of marbles with my hand (three were visible) and asked Brenda to count all of the marbles after telling her that there were six under my hand. She first counted my five fingers saying "1-2-3-4-5" synchronous with touching each one and then counted the three visible marbles "6-7-8". After I then said that I had six marbles in my hand, Brenda said "I don't see no six!"

Obviously, my concept of number differed from whatever meaning Brenda gave to "four" and "six". How could I possibly understand what her meanings might be like? The "trick" is, as Cobb and Wheatley (1987) said, "to develop and understanding of children's mathematics so that their actions can be seen as rational and sensible. The focus of a conceptual analysis is therefore on children's meanings" (p.2). But what would keep us from assuming Brenda had a concept of number (she did count perceptual items) and then try to imagine what we have to give up in, say, Hausdorff's cardinal number theory so that the remaining components would seem compatible with what Brenda does when solving tasks? This might seem to be a reasonable way to proceed, but I believe it is a particularly unfortunate choice.

Refusing to admit that human beings construct number and accepting that number is innate, as do Gelman & Gallistel (1978), can only delude us as mathematics educators. We can then assume that the numerical meanings of young children are not unlike ours. A corollary would be to teach Brenda as if her received meanings somehow fit her teacher's intended meanings—as if she can make numerical sense of what her teacher says about numbers in the way her teacher does or in the way the authors of textbooks do.

Pluralities and collections

My interpretation of Brenda being unable to count perceptual items she could not see is based on von Glasersfeld's (1981) notion of a collection. Collections are experiential composite wholes composed of perceptual items. In von Glasersfeld's model, children early in life isolate co-occurring sensory-motor signals in the stream of experience by a process of unitizing—cutting discrete items out of the flow of experience. These discrete experiential items form the "things" from which object concepts are somehow abstracted. When a child can monitor using an object concept and tell the

difference between using it once and more than once in an experiential situation. Its repeated use produces a **plurality** of perceptual items. Let us say, for example, that a child recognizes a perceptual situation as an instantiation of the concept it has associated with the word "cup". The child may continue exploring its visual field, assimilating another combination of sensory signals, and then another. If the child keeps track that its concept of cup was repeatedly used, it could utter the word "cups". The cups would be bounded in the child's visual field if and when the child perceives the table on which they are arrayed as a background. In that case, the cups on the table form an experientially bounded plurality, or **collection**.

Von Glasersfeld's notion of collection is compatible with Piaget's (1937) model for the developmental process that results in the child's conception of objects as externalized, **permanent objects** that are considered to have an existence in their own space and time independent of the experiencing subject. Brenda could use her object concept associated with "marble" to recognize (assimilate) actual sensory-motor experiences as instances of the object. For Brenda to believe that there were marbles hidden, she would have to use her object concept to create a re-presentation[3] of a marble regardless of the absence or presence of perceptual signals. There was good indication that Brenda could use her object concept in both of these ways, as we would expect children six years of age to be able to do. She could create **figural** as well as **perceptual** items. But these figural items did not appear to be available to her as countable items.

Returning now to Brenda's meanings of "four" and "six", during our exchanges I would say **Brenda had no object concepts for the words**. I know from other interviews that Brenda could recognize triangular and linear arrangements of three items as "three" and any two items as "two" after seeing the arrangements only for a moment. When a square array of four dots were shown, however, she had to count the dots before she could say "four". There was no indication she could re-present an arrangement of four dots and count its re-presented elements[4]. Moreover, she did not recognize a row of four dots as "four" nor a domino five as "five". Without actually counting these arrangements, she could not say how many dots they contained. Her meanings of number words "four" and beyond had to be made by actually counting collections and did not exist independently of the counted collections. There were no **object concepts** for these words and they referred to the transitory experience of counting collections or to its results. She was aware of counting in much the same way she was of, say, skipping rope. Prior to the activity, skipping rope would not be a meaning of "three". But if "1-2-3" was coordinated with skips, that would be counting, and the experience would be a meaning of "three". Children like Brenda for whom collections must be available to establish units that can be counted are called **counters of perceptual unit items** (Steffe, von Glasersfeld, Richards

& Cobb, 1983). They know how to count but need a collection of marbles, beads, fingers, jumps of a rope, etc., in order to carry out the activity.

One of the first manifestations of independence from immediate perception in counting occurs when a collection of items can be counted even though it is not within the child's range of immediate perception or action. In this case, a child might attempt to count the items of a screened collection by coordinating the sequential production of visualized images of the perceptual items with the sequential production of number words. In this case, I would say the child counted figural unit items. When a child can count only perceptual unit items, I refer to their counting scheme as a **perceptual counting scheme** and when they can count figural unit items as well, I refer to it as a **figurative counting scheme**.

Awareness of plurality

The question remains why Brenda seemed to be aware of the hidden items but, crucially, could not count them. An **awareness of plurality** requires the production of a visualized image of a perceptual item along with its actual repetitions. It is an awareness of more than one perceptual unit item. The repetitions create a collection of perceptual **unit** items because the perceptual items are now considered from the point of view of their unitariness or wholeness. A **figural plurality** is the experiential results of repeatedly re-presenting a perceptual unit item in the absence of sensory material. When there is an experiential context that forms a boundary for the process of creating figural unit items, we might think of the figural unit items as a **figural collection**. In Brenda's case, the cloth and my hand formed such possible experiential contexts. But it is my belief that Brenda could not repeatedly re-present her object concepts and hence, could not make figural collections.

If this was the case, as her inability to re-present a collection of two perceptual items seems to indicate (Steffe & Cobb, 1988, p. 36), then we still have to explain the sense in which she was aware of the hidden items. In the explanation, it is my assumption that for objects to exist for a child in space and time independently of the experiencing child, the child must be able to re-present the material in an object concept. Otherwise, there would be nothing in the child's awareness that could "exist". I take this to be the sense in which Brenda was aware of the hidden items.

An **awareness of figural plurality** requires the re-presentation of a perceptual unit item along with its repeatability. Given that Brenda could re-present a perceptual unit item but not a collection of two perceptual unit items, there is reason to believe that she

did not repeatedly re-present a perceptual unit item--repeatability was not a property of her figural unit items. Creating figural collections must be more demanding for some children than we adults might expect it to be.

An awareness of plurality (perceptual or figural) should be thought of as being indispensable in activating counting and a lack of counting figural unit items is an indication of an inability to make figural collections. How Brenda constructed figural collections is an interesting story to which I now turn.

Brenda's use of finger patterns

The construction of figural collections involved finger patterns for Brenda. The following protocol is an excerpt from a teaching episode held with Brenda on the 19th of March of her first grade year.

Protocol 2

T: (Displays four squares). See those?

B: (Nods her head "yes").

T: (Hides the four squares and displays three squares). See those? (Hides the three squares).

B: (Simultaneously puts up four fingers) four. (Closes her hand and sequentially puts up the same four fingers) 1-2-3-4 (continues sequentially putting up fingers on her other hand) 5-6-7.

Brenda now seemed to have constructed object concepts to which "four" and "three" referred. The act of simultaneously putting up four fingers while uttering "four" and then closing her hand and sequentially putting up the same four fingers while uttering "1-2-3-4" (counting is a strong indication that Brenda re-presented the finger pattern and then counted the elements of the re-presented pattern. In other words, "four could refer to a re-presented unitary whole comprising figural unit items that could be sequentially isolated and counted. The fingers of her figurative finger pattern were **countable figural unit items**.

It is especially significant that Brenda could coordinate the implementation of a finger pattern for "three" with her utterances "5-6-7". This shows that "three" referred to a re-presented finger pattern whose elements could be coordinated with any three number words in sequence. She made similar coordinations using finger patterns

associated with "two", "four", and "five". Her ability to re-present these finger patterns and count them in a continuation of counting provides the necessary basis for considering them to be figural concepts of the involved number words. At no time did I observe Brenda constructing meaning for number words using the relation "one more than", a relation that she was yet to construct. Moreover, I found that she could establish a relation between two collections by counting the perceptual unit items of the collections or by physically pairing them one-to-one. But these relations were of the same nature as her transient meaning of number words and could not be used as foundational in constructing object concepts for number words.

Activation of counting. There is a fundamental issue suggested by protocol 2. Apart from Brenda's ability to re-present finger patterns, how can we explain that counting was now activated in protocol 2 where in protocol 1 it wasn't? I could detect nothing different in the social situation in the two protocols that would suggest that Brenda expected to count in protocol 2 but not in protocol 1. In fact, in protocol 1, I suggested to Brenda that she count, but no such suggestions were made in protocol 2.

The semantic connections that Brenda earlier established between perceptual finger patterns and the number words "two", "three", "four", and "five" were the results of counting. If a child counts the fingers of a finger pattern "1-2-3", connecting the counted finger pattern with the last word said is an act of pseudo-empirical abstraction (Piaget, 1980). The counting activity can be curtailed because the pattern embodies its results. I think of the counting acts as being **recorded** in the finger pattern, having been introduced into the finger pattern by the child's activity. If a finger pattern is then isolated on some future occasion, the assimilation could lead to the activation of the records which in turn activates the response "three" without intervening counting activity.

Reflecting on finger patterns. A finger pattern provides an opportunity for a child to construct the dual meaning of the result of counting--unitary and at the same time composite. The items of a finger pattern seem to co-occur in the child's perceptual field and the experiential pattern gives the child something to reflect on. Without this dual meaning, it would seem very implausible to me that a child would reflect on the results of counting and isolate a connection between the last word said when counting and the finger pattern. That Brenda did reflect on her perceptual finger patterns is clearly indicated in protocol 3 excerpted from a teaching episode held on the 10th of February.

Protocol 3

T: (Places three marbles in front of Brenda) how many marbles?

B: Three (immediately)

T: How many marbles do we have there (places four marbles in front of Brenda)?

B: (Touches each marble) 1-2-3-4

T: (Places paper cups over the marbles) how many marbles do we have altogether?

B: (Stares straight ahead for about 25 seconds with her hands resting in her lap) I hope its seven!

T: It's seven alright! How did you get that?

B: I counted on my fingers!

Reflection requires the reflecting subject to "hold an object still at a distance". During the 25 seconds that Brenda stared straight ahead, she displayed intense concentration and seemed to be aware of what she was doing with her fingers as they rested on her lap. She seemed to hold the finger patterns still in her tactual and kinesthetic fields and then count the elements of those patterns. Her perceptual finger patterns evidently provided her with composite wholes of which she was aware. The whole protocol indicates to me that **internalisation** of her finger patterns was at least beginning.

Internalisation of finger patterns

Internalization can lead to visualisation in any sensory modality. It is the process that results in the ability to re-present a sensory item without the relevant sensory signals being available in actual perception or in the ability to re-enact a motor activity without the presence of the kinesthetic signals from actual physical movement. Staring straight ahead while counting indicates to me that Brenda re-presented the visual component of her finger patterns while creating sensory material in her kinesthetic and tactual fields. Using the visual records of her finger pattern in re-presentation while enacting the kinesthetic and tactual records serves in the internalisation of the enacted records.

Establishing her finger patterns in her tactual and kinesthetic fields while visualising them in her visual field indicates to me that Brenda was beginning to internalise the kinesthetic component involved in simultaneously putting up fingers.

Likewise, counting her fingers while staring straight ahead indicates to me that she was beginning to internalise moving individual fingers when counting. Protocol 2 more or less indicates a completion of the internalisation process and Brenda could now establish specific figural collections—her finger patterns. Her finger patterns were now **object concepts** connected to her number words 'one' through 'five' and she now had countable figural items available to count—her fingers.

Sequentially putting up fingers in protocol 3 indicates to me that **repeatability** was becoming a property of her figural items. Using her finger patterns circumvented the necessity of establishing repeatability as a property of more general figural unit items to create figural collections because she simultaneously put up fingers to establish a finger pattern. Nevertheless, the action that was introduced on the sensory-motor level by counting the fingers of her perceptual finger patterns and by the covert finger movements like those that must have occurred in protocol 3 **was implicit** in the representational finger patterns. It was made **explicit** when Brenda sequentially put up fingers synchronous with uttering number words to instantiate the finger patterns. Brenda **externalised** what had been previously **internalised** and her sensory-motor finger patterns established by counting reflected her object concepts.

On the 9th of May Brenda's solution of a task indicates a more complete externalisation of the motor acts implicit in her finger patterns. After the teacher presented the problem 'You have 13 dolls and I have four dolls. How many dolls do we both have?' Brenda asked if she could count and then sequentially put up all ten fingers, closed one hand, and sequentially put up three fingers for a second time while synchronously uttering '1-2-...-13'. She now seemed to focus on the activity of counting rather than on her global perception of the items she had established by counting (i.e., a finger pattern). This gave Brenda greater generative power because she was no longer stymied when she ran out of fingers. In particular, Brenda recounted three fingers when she established the collection of 13 fingers. As this collection could not be bounded by scanning visually, I infer that it was bounded by the beginning and end of the activity she performed while creating it. This is important because it indicates that she transcended visual perception in establishing the result of counting. This led to the establishment of sophisticated finger patterns.

Sophisticated finger patterns

A second issue that emerges from protocol 2 is whether her finger patterns were now numerical rather than only figurative[9]. One indication of their figurative nature

was their specificity. "Four", e.g., referred to the four fingers of either of her hands excluding her thumbs and "five" referred only to an open hand. In protocol 3, "three" referred to a finger pattern she completed on her remaining hand after completing a finger pattern for "four" using the index, middle, ring, and little fingers of her other hand. She did not use her thumb and two fingers of her remaining hand to complete a finger pattern for "three"--she did not split her finger pattern for "three" between two hands.

A second indication that her finger patterns were figurative rather than numerical was the necessity for Brenda to create an experiential meaning as well as a figurative meaning for number words, as we see in protocols 4 and 5. Brenda had established perceptual finger patterns for the number words "six" through "ten" prior the 19th of March, the time of protocol 3. She constructed these finger patterns along with sophisticated finger patterns for the number words "eleven" through "fifteen" as figurative concepts by the 7th of November of her second grade year. These sophisticated finger patterns were a complete surprise to me.

Protocol 4

T: (Covers eleven marbles with a cloth and then places three more on top of the cloth) now there are three more.

B: Eleven (simultaneously puts up five fingers on her left hand and one on her right hand to indicate "eleven"). 1-2-3 (sequentially puts up three of her remaining fingers) fourteen!

The patterns for "eleven" through "fifteen" involved taking an open hand as an abbreviated finger pattern for "ten". To clarify the status of Brenda's sophisticated finger patterns, I asked her to close her eyes when solving a task involving "eight" and "five".

Protocol 5

B: 1-2-3-4-5-6-7-8 (sequentially put up five fingers on her left hand and three on her right hand). 1-2 (sequentially puts up her remaining two fingers) 3-4-5 (sequentially wiggles three fingers of her left hand) thirteen!

Brenda said "thirteen" without her open right hand and three fingers of her left hand being in her visual field. This suggests to me that her results of operating were recognized in her tactual and kinesthetic fields using an internalized finger pattern for "thirteen"--an open hand and three fingers. Whether this finger pattern was to be called "eight" or "thirteen" was determined by context. These sophisticated finger patterns confirm that Brenda was indeed aware of the results of counting to "thirteen" by

sequentially putting up thirteen fingers, as I described earlier.

Dual meanings of number words

Brenda's finger patterns led to dual meanings of number words through "fifteen" by the first of February of her second year in school. In protocol 6, Brenda counted to find "fifteen plus nine".

Protocol 6

B: "1-2-3-...-15" (simultaneously puts up ten fingers) "16-17-...-24" (sequentially folds down nine fingers).

To give meaning to "fifteen", Brenda could simply utter the appropriate counting word sequence[6] or else put up ten fingers. In either case, she created an experiential meaning for "fifteen". Nevertheless, she had made progress because she had curtailed the coordinations involved in counting. This is what I call **counting verbal unit items**.

There was no indication during the two-year duration of the teaching experiment that Brenda **interiorized** her finger patterns, creating numerical finger patterns. So I now turn to exploring the meanings of number words of Jason, another participant of the same teaching experiment, in part to illustrate what I mean by the interiorization of patterns.

Jason's Construction of Meanings for Arithmetical Words

Before exploring interiorization, there is another issue that I left implicit in the discussion of figural collections in Brenda's case that I now make explicit. The issue is "Can children make figural collections using perceptual collections whose elements are not arranged in a pattern by the child and take their elements as countable?" To investigate the issue, I offer the following exchange with Jason. The protocol is excerpted from an interview that occurred on the 16th of October of Jason's first grade in school.

Protocol 7

T: (Presents two cloths that Jason takes as covering squares) there are six here and five here. How many squares are there altogether?

J: (Sequentially moves fingers on his left hand, moving his index finger twice) 1-2-3-4-5-6. (Continues sequentially moving fingers on his left hand) 7-8 (middle and ring finger) 9-10 (index and middle finger) 11-12 (moving no fingers).

This was the first time that I had worked with Jason, so I had no previous influence whatsoever on how he responded to my question. Jason had not established finger patterns as meanings of number words at this time in the teaching experiment expect for "two" and "ten". In fact, it wasn't until the 3rd of December of the same school year that I observed Jason use a finger pattern for "three".

Jason's meaning for "five" and "six"

Give that Jason had no finger pattern connected to "six", the interpretation of the first part of his counting episode becomes interesting. Did he have an object concept associated with "six" other than a finger pattern, or is there another way to explain his behavior? My choice is to attribute an awareness of a figural plurality of squares to him and an intention to count them. Given that he wanted to count, a lack of perceptual material to instantiate his concept, square, to make countable items could leave him in a state of activation, searching for something to count. It seems reasonable to me that Jason would "call up" records of his past counting acts in his search if "six" referred to counting the squares. That Jason was successful indicates to me that at least the auditory records of his past counting acts were **internalised**--they could be re-presented in the absence of actual sensory-motor material.

Figurative number word sequence. The "image of the standard number word sequence" refers to a more or less permanent record of the kinesthetic or auditory aspect of saying number words in the standard order (Steffe, et al., 1983, p. 26). When these records can be used in re-presentation, the number word sequence is internalized and I call it "figurative". Activating the image of the standard number word sequence can in turn activate instantiating the records, which means to say number words in sequence starting from "one". Being in a state of activation but having no squares in his visual field to count, Jason generated the sensory-motor items of putting up fingers as countable items.

To explain where the motor acts of putting up fingers come from, we have to look to when Jason counted perceptual unit items in other counting episodes. An **act of counting a perceptual unit item** is the more or less co-occurrence of uttering a counting word and instantiating an object concept, where the co-occurrence is often experientially determined by a guiding motor act—a finger movement. In that his intention was to count the squares and given that there was no perceptual material he could use to make a square, he enacted the guiding motor acts his number words signified and took **them** as countable items. Focusing attention on the guiding motor act between uttering a number word and instantiating an object concept when the there is no perceptual material that can be used in instantiation is what I mean by substituting a motor act for a figural unit item in counting.

So, I can see no reason to believe that Jason needed an object concept of "six" in order to count as he did, and it was not a necessary part of my explanation. I do believe that "six" referred to a figural plurality, but there was no necessity that he could "see" six individual figural unit items that appeared to co-occur in some pattern. The counting meaning of "six" alleviated that necessity. However, an internalized counting word sequence was essential for the search of something to count to be completed and for counting to be successfully actualized to create an experiential meaning for "six".

The explanation of the first part of Jason's counting episode is what I mean by a **counter of motor unit items**. After Brenda became able to sequentially put up fingers when counting to "thirteen", I also call her a counter of motor unit items because I believe by that time she became aware of a **figural plurality of fingers**. This is a more restricted meaning of being a counter of motor unit items than I explained in the case of Jason. Jason, in contrast to Brenda, could use almost any of his object concepts in creating figural pluralities and an awareness of those figural pluralities in part activated counting.

Jason's continuation of counting. I turn now to the second part of Jason's counting episode because it is so delightfully consistent with my explanation of the first part. After uttering "six" and putting up his index finger, he continued on uttering "7-8" while putting up his middle and ring finger, and then returned to putting up his index finger and his middle finger while uttering "9-10". At this point he seemed to be "lost in counting" and went on uttering two more number words "11-12". He seemed to not use an object concept connected to "five" to keep track of continuing to count beyond "six". His counting behavior indicates to me that he intended to count all of the hidden items and separating his continuation of counting from the first part further indicates to me that he was aware of an experiential separation in the items he was counting. An awareness

of the second part coupled with a counting meaning for "five" activated his continuation? He used his two-pattern in an attempt to keep track of his continuation of counting **but he did not monitor his use of the pattern, creating a pattern for "five" as a result of his activity.**

Monitoring a continuation of counting

It is interesting to speculate whether Jason's awareness of figural plurality was indicative of a number concept. In other words, should we consider Jason's result of counting to "six" to be an instantiation of a numerical concept for "six"? Although a counter argument has been presented, it is certainly a possible interpretation and I can imagine someone making it because there was a time when I interpreted equivalent counting behavior in that way (Steffe, Richards & von Glasersfeld, 1979). Since that time I have become aware of a more essential criterion for imputing numerical concepts to a child. Had Jason monitored his continuation of counting, creating a concept for "five" on the spot, then I would be willing to attribute the operations to him that are necessary for children to make numbers.

On the 18th of March of Jason's first grade in school, he did monitor a continuation of counting in such a way that indicated to me that he was in control of his actions in the counting episode. As early as January of his first grade in school, he had established linear and domino patterns for the number words through "six" as figurative concepts. Jason's teacher presented him with a "missing items" task, which he interpreted in his own way.

Protocol 8

T: (Places a cloth in front of Jason) **see those chocolate cookies under there (the teacher and Jason were only pretending)? Put the number on the cloth that shows how many you would like to put under the cloth (Jason puts "8" on top of the cloth). The teacher lifts an adjacent cloth) see those chocolate cookies under there?**

J: Uh huh (no).

T: **Well, let's put some under there. Now, there are ten cookies under both cloths (places the numeral "10" immediately above both cloths). How many are under here (the adjacent cloth)?**

T: (Touches the cloth with the numerical "8" on it eight times) 1-2-3-4-5-6-7-8. (Continues touching the other cloth as if it hid ten cookies) 9-10-11-12. (completes a row of four points of contact and then continues touching the cloth immediately beneath the completed row) 13-14-15-16. (Looks up at the teacher while saying "sixteen" and then continues touching the cloth immediately

beneath the two completed rows, continuing to look at the teacher] 17-18 [touches the cloth emphatically when saying "eighteen" indicating that he was done]

The crucial indicators that Jason monitored his continuation of counting in that he looked intently at the cloth he took as covering ten "cookies" as he completed two rows of four and then changed from looking intently at the cloth in order to recognize the successive patterns to looking at his teacher. This indicates that he reflected on his counting activity and "took stock" of where he was at as he was counting. He did not seek nor did he receive nonverbal cues that would indicate to him when to stop counting. He behaved in such a way that I have to attribute to him the operations necessary to create numerical patterns on the spot.

Jason could make numerical finger patterns as well as numerical spatial patterns. In the same teaching episode, his teacher placed "8" on one cloth as in protocol 7 and "12" above both cloths. Jason sequentially put up fingers synchronous with uttering "1-2-3-4-5-6-7-8" nd then continued, putting up two remaining fingers on his right hand (his thumb and little finger) while uttering "9-10" and then moved his index and middle finger of his right hand while uttering "11-12". Upon de completion of counting, he was left with two open hands. Nevertheless, after about five seconds, he said "four!" He obviously separated his first eight counting acts from those he subsequently performed to count to "twelve" in a review of the results of counting and recognized a finger pattern--four. As there were no explicit perceptual records in his continuation of counting using his unitising operation. In this way, he created what I call numerical finger patterns. This was a remarkable achievement for Jason at this time in the teaching experiment and constituted the first time I observed him solving a missing items task "correctly". I never observed Brenda monitor her continuations of counting in the way Jason did. I have to stress that Jason had *recently* monitored his counting activity without suggestion. Nevertheless, the processes Jason used to establish meaning for "ten" are not even close to those implied by the relation "one more than", which he was yet to construct.

Figurative counting schemes revisited. I can now make the notion of a figurative counting scheme a little more explicit. I have already said that when a child can create and count the items of a figurative collection, their counting scheme is a figurative scheme. Figurative counting schemes include those cases where the child can create and count motor acts as did Jason or can simply utter counting words in sequence. In the latter case, the child has created an internalized sequence of counting acts. The involved number word sequence is internalized as well as the kinesthetic, visual, and auditory aspects of counting. Because the kinesthetic aspect is also internalised, the child's can

utter number words in sequence with no other accompanying motor acts, where the number words signify other countable items. In these cases (counting motor and verbal unit items), the experience of counting is bounded by its beginning and end and this experience constitutes the meaning of the last number word said when counting. There was certainly no indication in protocols 2, 3, and 6 for Brenda and protocol 7 for Jason that these children took the items of the figurative patterns or collections as one thing prior to counting because they always started to count from "one"[6]. There also was no indication in the protocols that the children took the items of the counted collection as one thing.

Interiorization of the counting schema

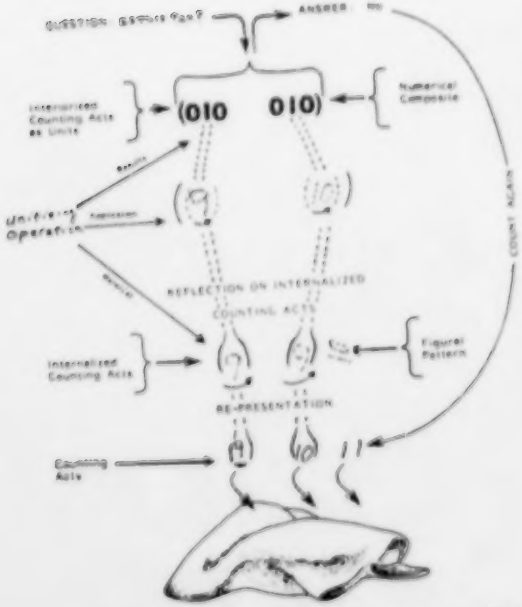
When Jason was producing a pattern consisting of the records of his points of contact of his finger on a cloth in protocol 8, he had to establish what pattern he produced after each touch because there were no visible traces of his points of contact. The act of simply recognizing a row of dots by saying a number word does not require an intentional monitoring of the activity that produces the pattern. In producing the patterns, he had to recreate his acts of counting for the simple reason there were no visual traces of counting. I have diagrammed the situation in Figure 1 after Jason had counted "0-10", when it was his intention to continue to count until completing a pattern for "ten".

Re-presentation alone is not sufficient to explain Jason's monitoring behavior because in protocols 2, 3, 6 and 7, Brenda and Jason could use figurative patterns to keep track of a continuation of counting but could not create patterns when intentionally monitoring counting. I did not observe Brenda create a pattern in the activity of continuing to count at any time I taught her over the two years of the teaching experiment.

To monitor counting activity intentionally as did Jason in protocol 8, there must be an explicit awareness of a figurative pattern, which is to say the child must reflect on the figurative pattern. To explain how Jason was explicitly aware of the figurative patterns he used to take stock of where he was at after completing each counting act, my hypothesis is that he applied his unitizing operation to the elements of the figurative patterns. "Running through" or reprocessing the elements of the patterns strips the figurative counting acts of their sensory-motor quality and creates a sequence of abstract unit items that contain records of the counting acts (abstract discreteness). The sequence of abstract unit items is what I call a numerical pattern or composite--a pair of

Figure 1

Jason's Initial Interiorization of Counting



NOTE: Numerals rather than number words are used to simplify the figure.

↳ Denotes pointing acts.

interiorised counting acts. The concept that Jason finally created for 'ten', was, in my model, a numerical pattern that contained the records of counting '9-10-11-12-13-14-15-16-17-18'.

A reorganisation of Jason's counting scheme. By the 31st of May of Jason's first grade in school, he could use his counting scheme in ways that were previously not possible.

Protocol 9

T: (After Jason places 16 poker chips in a cup) take out four.

J: (Takes out four)

T: How many are left in there?

J: (Places his hands on his lap and concentrates on manipulating the poker chips. The teacher cannot see what Jason is doing) there's twelve in there.

T: Tell me how you did that!

J: (Places the poker chips on the table one at a time) 16-15-14-13--there's twelve in there.

Protocol 10

J: (Puts 14 checkers into a cup).

T: (Puts more checkers into another cup) altogether there are twenty. How many are in here?

J: I was thinking there was six!

T: Ok. (Takes three out and pours the rest into Jason's cup, which contains fourteen) there are seventeen in there. How many did I put in?

J: 15-16-17--three!

The change in Jason's use of his counting scheme implies an underlying reorganisation of the scheme. Because I had never observed him use it in these ways prior to this teaching episode, it is not too much to say that his counting scheme had undergone a metamorphosis and was now an interiorised scheme.

An interiorised counting act is an abstract unit item that contains records of an internalised counting act. I too can be re-presented by re-presenting what the records point to. So, an interiorised sequence of counting acts is a sequence of abstract unit item that contain records of the internalised sequence of counting acts that served as

material in its construction. I call such an interiorized sequence of counting acts a **number sequence**. In the case of the initial construction, "initial number sequence" is used. By a **verbal number** sequence, I mean a re-presentation or re-enactment of a number sequence using the auditory records--an operative counting word sequence. The lexical items of a verbal number sequence symbolize abstract unit items as well as counting acts.

From Jason's perspective, he still had only one counting word sequence. Nevertheless, he could certainly use it differently now that it had been interiorized. From protocols 9 and 10, I infer that Jason's number words could symbolize initial segments of his verbal number sequence. "Sixteen", for example, seemed to symbolize the verbal number sequence starting with "sixteen" and proceeding down to "one". He partitioned this sequence into "16-15-14-13" and "12" by counting backward, where "twelve" symbolized the rest of the sequence. The inference is strengthened by Jason's taking counting to "fourteen" as a given when he counted up to "seventeen". Moreover, he made an uncannily accurate estimate of the numerosity[9] of the checker placed with the original fourteen. I infer that his estimate "six" referred to the counting acts he would perform were he to start counting with "fourteen" and proceed on to "twenty".

Monitoring as feedback. The accommodation of Jason's counting scheme that engendered his reorganization of counting was monitoring a continuation of counting (cf. protocol 6). To explain what I mean by monitoring, I refer again to Figure 1. Application of the unitizing operation to re-presented countings acts--"9, 10" in the figure--disembeds the involved counting acts from the rest of the internalized counting scheme and records them at the level of interiorization. Jason, intending to continue on counting, could now take the interiorized counting acts as belonging to the items he was counting; to the **countable** items. The application of the unitizing operation thus made it possible for the counted items to **feed back** into the countable items and this is what I mean by monitoring counting in this case. Simply put, the child reconstitutes the counted items as countable items. Jason's goal to continue on counting ten times sustained this feedback system until the goal was reached. That Jason knew when he had reached "ten" is conformation that he reconstituted his counted items as being countable.

I do not believe that Jason "double counted" in the sense of saying "9 is one, etc." Being involved in counting, Jason created two four patterns (numerical patterns) and he knew that two four patterns make an eight pattern. This was intuitive because it was based on figurative patterns, a level below where he was operating. That two four patterns make an eight pattern was implicit in his operating as was the completion of a

pattern for "ten". Using figurative patterns, "ten" referred to two four patterns and a two pattern. The numerical patterns, being a focus of his attention, were explicit.

Autoregulation of monitoring. The metamorphosis of his counting scheme in protocols 9 and 10 can be explained by **autoregulation of the monitoring activity**. I hypothesize that a perturbation was created by part of his counting word sequence being recorded at the level of interiorisation and the rest of it being only internalized. It existed on two levels. I further hypothesize that this perturbation was neutralized by autoregulation of the monitoring activity. There are no assumptions made about how many times the unitizing operation was reapplied or how many adjacent number words might have served as material for a particular application. The only assumption that I make is that the unitizing operation was reapplied enough times to interiorize his figurative number word sequence. The completion of the interiorisation process was indicated when Jason used his verbal number sequence in assimilation and solution of missing items tasks in protocols 9 and 10.

The counting scheme as a numerical concept. Jason's number words through the teens now symbolized initial segments of his verbal number sequence. These symbolized segments in turn symbolized sequences of abstract unit items as well as the activity and results of counting from, say, "one" up to and including "sixteen". In protocol 10, for example, Jason took "sixteen" as symbolizing the numerosity of any sequence of perceptual unit items he could create and count using his verbal number sequence. He could also re-present the symbolized initial segment in either the forward or backward direction.

An initial number sequence should be thought of as an elaborated numerical pattern. To use it means to instantiate the records that compose it, or more simply, to count. When counting, the child is aware of the counting acts inside of the beginning and end of counting. After counting, the child is aware of what is inside of the boundaries of counting but does not take the bounded results of counting as one thing, although it is a connected activity for the child. A lexical item of the initial number sequence symbolizes these operations and, as an item of a sequence, it refers to the lexical items up to and including the given lexical item.

An **extensive** meaning of "sixteen" was provided by the symbolized initial segment of his verbal number sequence and the counting acts symbolized by this initial segment constituted an intensive meaning of "sixteen" (Thompson, 1982). Because of the symbolizing function of Jason's number words, it could be said that, from Jason's perspective, the meaning of number words was an intention to act (Van Engen, 1949).

The possibility of acting provided a dynamic quality to Jason's numerical meanings and it was a source of his confidence that he could solve a wide variety of numerical problems. We can see how important these possibilities were in protocols 9 and 10^{10}.

I regard Jason's initial number sequence as a numerical counting scheme. As activity, counting fits Piaget's (1980) characterisation of scheme: "All action that is repeatable or generalized through application to new objects engenders ... a 'scheme' (p. 24). But to view counting as simply activity provides no insight into why children count nor what the result of the activity might be for children. Toward such an end, von Glasersfeld (1980) separated Piaget's notion of scheme into three parts. First is the child's recognition of an experiential situation as one that has been experienced before; second is the specific activity the child has come to associate with the situation; and third is the result that the child has come to expect of the activity in the situation. Von Glasersfeld's emphasis on experiential situations is clarified by a comment Piaget (1964) once made about a stimulus in a critique of associationism:

> A stimulus is really a stimulus only when it is assimilated into a structure and it is this structure which sets off the response. Consequently, it is not an exaggeration to say that the response is there first (p. 15).

After Jason's counting scheme was interiorized, it definitely was not an exaggeration to say that his initial number sequence contained counting. A number word, say "sixteen", had been involved in the process of interiorization of counting and it therefore could **symbolize** the operations that were involved in that process and their result, a sequence of abstract unit items that contained the records of counting. So, the activity of counting had become an aspect of the first part of his counting scheme.

Engendering and metamorphic accommodations. The interiorization of the counting scheme also emerged in the context of patterns in the cases of Tyrone and Scenetra, Jason's cohorts in the teaching experiment. Figurative concepts of number words seemed to play a crucial role in the construction of the initial number sequence for all three children. In retrospect, it isn't surprising that interiorization of the counting scheme emerged in the context of figurative patterns, because a pattern provides children with an experiential unity whose elements seem to co-occur; an immediate apprehension of plurality and unity.

Alone, however, figurative concepts of number words would never propel a child forward into the realm of number. It takes reprocessing them using the unitizing operation to create numerical patterns. But without perturbations being induced in the use of their counting scheme, I cannot imagine how children would construct an initial number sequence. For example, before Tyrone reorganised counting, he lost track of

counting over a cloth covering the second hidden portion of a collection because he did not recognize touching the cloth five times with his finger. He started over again without suggestion from his teacher and monitored the nonvisual records of his counting acts, creating a pattern for "five" on the spot (Steffe, 1988, p. 153). This was a solid indication that he was aware of an uncertainty in the results of counting.

Scenetra used her unitizing operation in the context of patterns to neutralize perturbations she experienced in assimilation. In the context of finding how many squares were in a collection, where five were hidden and six were visible, Scenetra counted the visible squares and then tried to use her perceptual finger patterns. Not having enough fingers created a perturbation she neutralized by changing her scheme of operating from finger patterns to counting. I inferred that she reprocessed visible squares that she had previously counted using her unitizing operation because, in her words, she "put them in my head" and counted on from "six" five more times. The independence she exemplified in switching from one schema to another indicates a deliberate choice and reflection on her actions. I believe both her choice and reflection were made possible by her creating numerical composites using her figurative pattern (Steffe, 1988, p. 178).

Although I have not provided a protocol for it, Jason also reinitialized his counting scheme as the result of a perturbation (Steffe & Cobb, 1988). In protocol 8, he monitored his counting acts, and in the counting activity immediately following protocol 8, he reviewed his records of past counting activity. In fact, all three children could monitor their current counting actions or review their past counting actions. These modifications turned out to be **engendering accommodations**[11]. The **metamorphic accommodations**[12] to which they lead in the cases of Tyrone and Scenetra were manifest by the 8th of January and the 5th of March, respectively. These were the dates I first observed reorganizations in the two children's counting schemes that were on a par with reorganizations Jason made in protocols 9 and 10.

The period of the initial number sequence

One of the most significant but surprising results of the teaching experiment was that approximately eight months elapsed before any of the three children used their initial number sequence in such a way that indicated another fundamental reorganization. During this period, a novel use of the unitizing operation appeared that is called the **uniting operation**. This operation has been identified by Menninger (1969) in a study of the cultural history of numbers. Menninger commented that in two as

unity, "we experience the very essence of number more intensely than in other numbers, that essence being to bind many together into one, to equate plurality and unity" (p. 13). This is compatible with Brouwer's (1913, p. 65) analysis of the intuition of mathematics as being the bare "two-oneness".

The issue of whether two is a property of two perceptual items is critical. According to Mennınger, two cannot be a property of two perceptual items because it cannot be a property of a single perceptual item and, thus, two cannot be found in nature. Rather, it takes an operation of the mind that binds distinct unitary items together into a unity and two can be a property of that composite unity.

> The intellect ... does not find numbers but makes them. It considers different things, each distinct in itself, and intentionally unites them in thought (Caramuel, 1970, p. 44). Trans. by E. von Glasersfeld.

To make a unit of units, a child can reprocess a collection of perceptual unit items using the unitizing operation and then reprocesses those results using the unitizing operation as indicated by Caramuel. From here on, when I say that a child uses the uniting operation, I refer to this process. The material of operating, however, does not have to be a collection of perceptual unit items. It could be, say, re-presented counting acts. In some cases, reprocessing a collection of perceptual unit items is symbolized. In other cases, a child might re-present the material of a sequence of abstract unit items that were created in the past by using the unitizing operation. In these cases, I say that the involved sequence of abstract unit items are taken as a given by the child. This can occur especially in the case of the initial number sequence because a lexical item of the sequence can symbolize counting as well as the verbal number sequence of which it is the last element.

Adding and subtracting schemes. Toward the last part of the period of the initial number sequence, the children often took the experiential results of counting as material of application of their uniting operation. Moreover, they constructed adding and subtracting schemes using their initial number sequence. Protocol 11 is excerpted from a teaching episode on the 14th of December of Jason's second grade.

Protocol 11

T: (Places "17 - 9" in front of Jason) what would that be?

J: 17-16-15-14-13-12-11-10 (in synchrony with putting up fingers). That would be (pauses) ... nine!

It would be very unlikely that Jason would say "nine" without taking the results of counting as one thing. Although he could be trained to do that, the independent way in which he said "nine" along with that fact that he did say 8 indicates to me that he took the results of counting as one thing--as material of his uniting operation. He could then "rise above" his activity and "see" his verbal number sequence 17, 16, ..., 2, 1 as being separated into two parts--into (17, 16, ..., 10)[13] and into the other part which was symbolized by "nine".

It is interesting to speculate on what went on before Jason counted in protocol 11--on his subtraction concept. I look to Jason's subtracting acts (his counting acts) for insight because they were not simply connected to "17 - 8" as an appropriate movement in square dancing is to "do-si-do". Based on his counting acts, his subtraction concept seems to be best understood as a sequence of possible actions--interiorized actions--that he isolated in the past when he used his counting scheme. Although I do not trace the construction of his subtraction concept, we do see its initial form in protocol 9[14]. There, he counted backward only after he physically removed the four poker chips to find where he was at before he counted the four he removed. Counting backward would not have been activated by expressions like "16-4". It did provide a possible experience for the necessary abstractions in constructing a subtraction concept. In contrast, in protocol 11, "17 - 8" symbolized subtracting actions.

So, rather than physically remove eight items from a collection of 17, he used the elements of a numerical finger pattern as symbolizing the items being removed. Using this concept of "eight", he counted "17", then "16", etc., "off from" the backward verbal number sequence. The lexical items of his verbal number sequence were his countable items and he "counted" them by recording each word said by putting up a finger. Putting up a finger carried the force of uttering a number word of the sequence "1-2-...-8". Functionally, he was double counting but he could not yet explicitly double count. What would be explicit later was now only implicit. Taking the results of counting--the elements of a counted finger pattern--as one thing certainly constituted a reinteriorization of part of his counting scheme as well as an accommodation of counting to reach his goal[15]. After counting, he had a solid basis for establishing numerical connections among the three principal number words.

Before counting, Jason's intention seemed to be to separate the backward segment of his verbal number sequence symbolized by "17" into two parts, the first consisting of eight lexical items and the second as an initial segment of yet unknown numerosity. But I take this goal to be only implicit in his actions. In other words, I don't think he consciously made plans prior to counting. It just appeared to him as the thing to do

without knowing why or without having alternatives. The 'goal' I alluded to above was created by the activation of his program of subtracting actions and his anticipation of what the result would be. Carrying out the actions was one way he could satisfy his goal. It is worth noting that his estimate 'six' shows that he could anticipate what the results of counting would be if it were to be carried out, which indicates awareness of results of counting but not necessarily of why he counted as he did.

Whatever the terms 'addition', 'subtraction', and 'missing addend' might mean to us as adults, the three children interpreted our problem situations in terms of their initial number sequences--as counting problems. Counting was symbolized by arithmetical words. For example, it could be the intention of the children to count to 'twenty-one', say, to find how many of a collection of 21 blocks were hidden. If the children counted the visible items, they could apply the uniting operation to the counted items and then continue to count to 'twenty-one', whereupon they could once again apply the uniting operation to their records of continuing to count, creating a numerical whole whose numerosity could be established. These are what I call **sequential uniting operations**. If the children were simply told how many items were visible, that number word could symbolize an uniting operation and they could in this case again continue to count. This symbolizing function of number words permitted the children to appear to be able to perform more sophisticated operations than sequential uniting operations. It is important to note that sequential uniting operations were carried out during rather than before problem solution. In this sense, the children were in the process of constructing more sophisticated addition and subtraction schemes than I have explained.

The major insight that the children could at most make sequential uniting operations in activity provides a powerful explanation for certain observations that are not otherwise explicable. For example, not one of the three children could strategically find the pairs of numbers whose sum was, say, ten. They could find certain pairs, but they could not use a given pair to generate the next pair. And this was a general characteristic because the children solved each situation independently of prior related situations. Also, they could not understand subtraction as the inversion of addition nor could they construct a meaning of 'ten' as one ten prior to acting. This restriction in their meaning of 'ten' prohibited them from constructing counting by ten as an anticipatory scheme and from using this scheme to establish how many units of ten could be made using a particular number sequence or collection. For example, they could not independently count by ten, keeping track of how many times they counted, to find how many units of ten could be made from 59 blocks. They learned linguistic rules in their classroom for separating '59' into 'five tens and nine ones', but that was unrelated

to their number sequence.

The explicitly and tacitly named number sentences

On the 29th of March of Jason's second grade in school, the way Jason could solve problems changed dramatically. One strategy was to estimate the missing addend, add the estimate to the first addend, and check the result to see if it was the sum.

Protocol 12

T: (Places the expression "27 + ___ = 36" in front of Jason) we have twenty-seven and some more and that is thirty-six.

J: Twenty-seven—(pause of about 20 seconds). Let me see—(another pause)—twenty-seven plus seven—it's nine more!

T: That's really good! Is there another way to solve that one?

J: Uh-uh (no).

T: Could you do it by counting backwards?

J: (Sequentially puts up fingers) 36-35-...-27. Nine.

Jason seemed to view 27 as a number in its own right, as well as a part of 36 because he did not simply make an estimate and quit. Rather, after he made an estimate, he saw it as a possibility rather than as the answer. After estimating the missing number to be 7, he added it to 27, continued to 36, and then added the continuation to 7. These operations indicate that he used the units the numerals referred to as one thing, as well as composites. Moreover, 27 and 36 were related, in that 27 was a part that he had disembedded from 36. These claims are confirmed by the ease with which he counted from 36 down to 27. The teacher also tried his best to present "36 - 9 = ____" as a novel task immediately afterward.

Protocol 13

T: I am going to give you another one. This time we are going to have something to take away (presents the sentence).

J: (Immediately) Twenty-seven.

T: How did you know?

J: Because ... we just did it!

Jason clearly viewed subtraction as the inversion of addition. In fact, the two problems involved identical parts of the same whole, and he merely had to recover the known missing part. Addition and subtraction were now explainable as reversible part-whole operations. "Thirty-six" referred to [1, 2, 3, ..., 36], where the parentheses are used to connote a **unit** containing a number sequence. "Twenty-seven" referred to a segment of thirty-six, to [1, 2, 3, 4, ..., 27], and it was Jason's intention to find the numerosity of the remainder of this segment in thirty-six--[28, 29, 30, 31, ..., 36]. I believe he could disembed the two parts of thirty-six from thirty-six while leaving them in thirty-six (he did not destroy the whole by focusing his attention on the parts). Thirty-six was still "there" beside the parts (it was at least symbolized by "36"). In establishing the numerosity of the remainder, he could move back and forth between the parts and the whole, establishing new parts and combining them with the old parts to yield partial wholes. These operations provided him with great flexibility and were a complex form of the operations he had to carry out to understand that twenty-seven **was** the **answer** to "36 - 9 = ____" without counting.

When comparing Jason's methods of operating with numerical symbols in protocol 13 with his methods in protocols 9 and 10, it is as if he was a different child. His number sequence had undergone another metamorphosis. It is quite complex to explain because there was an intermediate step when he still operated sequentially. I call his number sequence in protocols 12 and 13 his **explicitly nested number sequence** because any word of the sequence symbolized a corresponding initial segment that he saw in two ways--as a composite and as a unit. A second critical characteristic was that he could, at one and the same time, view the composite unit as part of the sequence as well as a number in its own right. His number sequence was now **graduated**.

The tacitly nested number sequence. The initial number sequences were not inclusive in the sense that one is included in two, two is included in three, three is included in four, etc., as Piaget's model of the development of number would have us believe (Sinclair, 1971). The initial number sequence is just that for children--a sequence. The children were yet to construct **nested** number sequences. A number word was yet to symbolize the number sequence up to an including that number word as one thing. It could symbolize the individual number words in sequence, but it was yet to symbolize a unit containing that **sequence**. The children still had not constructed a "one more than" relation. They could isolate the number words following and preceding a given number word, but a relation of precedence and "one more than" are quite different because the latter implies order with inclusion, whereas the former implies order without inclusion.

To explain how the children may have constructed a nested number sequence, I turn to the sequential uniting operations that I illustrated in protocol 11. They exemplify the emergence of the uniting operation in the context of the initial number sequence. The technique of hiding part of a collection of items and then asking the children to find how many were hidden contributed to the application of the uniting operation to segments of the initial number sequence. However, performing sequential uniting operations is not the same thing as having these operations symbolized. In fact, the difference is profound. For example, after Ikenetra constructed the tacitly nested number sequence, she independently made a choice between using counting-downs-to and counting-off-from the first time she was observed using the former to solve a subtractive situation[16], and we had not worked on counting-down-to in subtraction with her. The solution episode was quite dramatic and occurred in the same way for Tyrone. His use of counting-down-to to solve a subtraction situation was a surprise because we had not worked with him in solving a subtraction situation in that way.

There was reason to believe that Jason reorganized his initial number sequence by the 22nd of February of his second grade, about two months after the teaching episode of protocol 11. We can see the progress Jason made in protocols 14 and 15 excerpted from a teaching experiment on the 15th of February of his second grade. Just before protocol 14, Jason had counted-on to solve "19 + 6 = ____".

Protocol 14

T: (Makes "19 + 6 = 25" using felt numerals) oh, Jason, can you find any more numbers that make twenty-five?

J: Twenty plus five.

T: Any more?

J: Eighteen plus seven!

Jason then proceeded to say the appropriate expressions to "thirteen plus twelve" after being asked by the teacher each time. The most striking feature of his behavior was the seemingly effortless way in which he generated an appropriate sequence of expressions. This is one of the first times he was observed using a strategy to generate possible addends whose sum was a fixed number. It is called a local compensation strategy because Jason used known sums to generate other sums by adding and subtracting one, starting with a sum he had just found. The second noticeable advance was that Jason could coordinate two number sequences.

Protocol 15

T: (Makes "31 + 6 = 37" to ... felt numerals) can you think of some other problems which are like that one, but where you keep thirty-one the same?

J: No, I can't.

T: What would it be if it was seven instead of six?

J: Thirty-one plus seven equals thirty-eight.

T: Can you tell me some more?

J: Thirty-one plus nine is forty. ... thirty-one plus fifteen is forty-six.

T: How far do you think you could keep going like that?

J: 'Till one hundred.

The strategies that Jason was capable of using while he was in the period of his initial number sequence were not iterative. By the 14th of December of his second grade, he could find the sum of nine and eight by using eighteen as the sum of nine and nine and decreasing it by one, but he did not iterate the strategy. Iterative strategies did not emerge until the 19th of February. In fact, I did not observe him use any strategy until the 14th of December of his second grade.

To explain his shift to iterative strategies requires postulation of a new use of the uniting operation—progressive uniting operations. He could now take the results of using his uniting operation as material of its application. As a consequence, a number word was now a symbol for a unit containing a number sequence from one up to and including that number word (or any other segment of a number sequence of specific numerosity indicated by the number word).

In protocol 15, Jason could create new unit items by performing a counting act and unite them with what preceded. For the first time, Jason's counting activity could be described as did Sinclair (1971) when explaining Piaget's model of the construction of number. In effect, Jason doubled counted "7 is 38, 8 is 39, 9 is 40, ..., 15 is 46", where the constant addend "31" was understood. I take Jason's words of the two involved number sequences as being his symbols for the progressive uniting operations that he could carry out. "Seven", for example, could be united with the counting acts implied by "six", but I can see no reason to believe that was actually done by Jason as he counted.

Autoregulation as a process of construction. Autoregulation can be used to explain how procedural accommodations like those illustrated in protocol 11 could lead to

reorganization of the initial number sequence. In protocol 11, Jason's concept for "eight" had been activated and served as a template for monitoring his further activity. The coordinated sensory signals generated by uttering a number word and the act of putting up a finger fed back into his template and he could combine them into discrete experiential items, one after the other, until he filled the slots that constituted his numerical finger pattern. Without this feedback, there would be no possibility for me to explain how Jason reorganized his counting scheme.

When Jason united the elements of his experiential finger pattern together into a unitary whole, this left the experiential finger pattern "behind" and created an abstract unit comprising a sequence of "slots" that contained the records of the experiential finger pattern. Because it was his numerical finger pattern that had been instantiated (his template had been "externalized" because of the feedback), this amounted to a modification in his concept for "eight". The word could now refer to what I call an "abstract composite unit".

There is no reason to believe that Jason took his finger pattern associated with "15-16-17" in protocol 10 as material for his uniting operation because he could simply recognize the finger pattern as "three". In protocol 11, however, a deliberate act was necessary to extract himself from the experience of creating a finger pattern because he had to take that activity as having been completed in order to focus his attention on the remaining number sequence. He had to "rise above" his experience of the finger pattern and focus his attention on the next lexical item in the backward sequence. Because this item ("nine") symbolized what was not spoken, there was a necessary element of reflection on what he was doing.

This might seem to be enough to reorganize his initial number sequence. But we have to remember that the involved number words were "17-16- ...-10" and they were involved in the context of his finger pattern. He had created a novel object concept for "eight" that contained these number words. "Eight" still referred to an initial segment of his initial number sequence. In so far as this initial segment was implicit in his instantiated finger pattern for "eight", he now had constructed an abstract composite unit for "eight" whose elements contained coordinated records of two segments of his verbal number sequence.

This is a reinteriorization of a segment of the initial number sequence, but the reinteriorization was not at a new level. It was still at the same level as the initial number sequence because the numerical composite for eight was first instantiated in experience and then the experiential results of operating were reprocessed, which does constitute a reinteriorization but not at a new level. What I mean can be understood by

formulating a possible way of operating that Jason may have used. Had Jason, after counting "17, 16", re-presented the two counting acts and then took the re-presented results as one thing, this would have re-interiorized the two counting acts at a new level and would have involved monitoring counting backward as I have already explained earlier. This provides the child with a possibility of constructing novel operations—progressive uniting operations. For example, after taking "17, 16" as counted items, if Jason re-presented these two counted items and then applied the uniting operation to them, this would create a unit of two countable items. Then uttering "15" (regardless of whether it is coordinated with putting up a finger) and re-presenting the three lexical items involves re-presenting the unit containing the records of saying "17, 16" and applying the uniting operation to what is in re-presentation. This creates the possibility of the child seeing "17, 16, 15" in tow ways—as a unit containing records of the three items, where the first two contain records of belonging to a unit created as a result of prior operating. These are records of progressive uniting operations created in operating. To create progressive uniting operations, the child operates on re-presented elements of the initial number sequence using the uniting operation.

Continuing until counting eight times creates a novel number sequence "1, 2, 3, 4, 5, 6, 7, 8" that contains the records of progressive uniting operations at a level of interiorization "above" the initial number sequence. This is a recurrence of the disembedding operation that engendered the initial number sequence, but in a more sophisticated form. Double counting and progressive uniting operations are introduced as novelties. These novelties engender the tacitly nested number sequence, a sequence of abstract unit items that contain records of counting acts, but with the additional feature that when the auditory records are used in re-presentation, the lexical items of the verbal number sequence symbolize progressive uniting operations.

As before, it is not plausible that a child would complete the construction of the tacitly nested number sequence in the context of solving problems like I have described above. I don't know how many such problem solutions it might take for a child to create a partial number sequence at a level superior to the initial number sequence and a feedback system from the initial number sequence to that partial sequence that is strong enough to sustain autoregulation of the disembedding operation until the process of reinteriorization is completed. But that is what it takes to construct the tacitly nested number sequence.

During the approximately two month period between the 14th of December and the 22nd of February, there were opportunities for Jason to solve problems in the way I have described (cf. Steffe, 1988, p. 209). Moreover, the process a child uses who is solving a

problem can only be inferred, so it is possible that reinteriorization of the initial number sequence occurred when I least expected it as a teacher. Although I have no estimate how many or which problems Jason may have solved, there were apparently enough to sustain the type of perturbations that I have explained and to activate autoregulation of the application of the uniting operation to re-presentations of his initial number sequence. Without appealing to autoregulation to complete the reinteriorization of his initial number sequence that had been initiated in experiential contexts, I could not explain the underlying reorganizations that led to his counting behavior in protocols 14 and 15.

The explicitly nested number sequence

Jason's period of the tactily nested number sequence was preparatory for the emergence of the explicitly nested number sequence illustrated in protocol 12. Experiential episodes like those illustrated in protocols 14 and 15 can only encourage the child to apply the uniting operation to figurative segments of the tactily nested number sequence, creating a unit that contains such segments. This again seemed to led to an internal reorganization of Jason's number sequence. The new operations allow a child to operate in a way that is no longer sequential. A number word can now symbolize a unit that contains a tactily nested sequence. The symbolized sequential operations can be carried out, of course, but the child finds new power in leaving them unexpressed.

An important result is that the inclusion relation that was implicit in the tactily nested number sequence is now made explicit. The child can "see" the inclusion relation in the reorganized number sequence, because it is symbolized by the sequence and can therefore intentionally disembed a segment of the explicitly nested number sequence from its inclusion in the containing sequence, and treat it as a unit in its own right. The child can also perform these operations using units for which there is no explicit symbol (like the remainder of 27 in 36). Moreover, the child can replace what was extracted, which is another way of saying the child does not destroy the records of the original number sequence by performing these operations. They can now have two number sequences side-by-side, and understand that one of them can be included in the other, whereas before, they could only coordinate two different number sequences. These sophisticated, reversible part-whole operations are critical to the mathematical progress of the child.

Final Comments

Uses of "implicit" and "explicit"

The words "implicit" and "explicit" were used in several ways in this chapter that can be made more general. First, I spoke of Brenda's finger movements as being implicit in her figurative finger patterns and of the movements being made explicit when she sequentially put up fingers when counting (cf. p. 13). I also said that Jason's counting acts "1, 2, 3, 4, 5, 6, 7, 8" were implicit in his numerical finger pattern associated with "eight" and could be made explicit by counting (cf. p. 37). More generally, I would say that Jason's counting acts were implicit in a sequence of abstract unit items. Applying the unitizing operation to sensory-motor or figurative items "strips" them of their sensory-motor material. Because there are records of the sensory-motor material that was "left behind" by the stripping action, what the records of the sensory-motor material that was "left behind" by the stripping action, what the records point to are implicit in the unit pattern that was "lifted" from the antecedent items. Although the discussion of Brenda's finger pattern did not involve the unitizing operation, it could be elaborated in that way.

On another occasion, I used "implicit" a little differently. When speaking of a subtracting scheme Jason had constructed, I spoke of a goal of the scheme as being implicit in his actions (cf. p. 31). There is definitely a stage in the construction of subtracting schemes and adding schemes where the schemes are interiorized and are available for use, but where the child has no awareness of addition and subtraction as operations. In this case, if the child assimilates a situation using their subtracting scheme, the three parts of the scheme are used in assimilation rather than just the first part because all three parts are contained in (or are implicit in) the first part. In Jason's case, "17 - 8" could symbolize the **result of counting** because the activity was recorded in the sequence of abstract unit items symbolized. Being aware of the possible results of acting, or anticipating acting, means that the response of the scheme has been activated but not actualized. The state of activation (or goal) constitutes a perturbation that can be neutralized by acting and this is why I said that the goal was implicit in Jason's subtracting actions. It could not be separated from the scheme; rather, it was made possible by the functioning scheme. For a goal to be explicit, it would have to refer to no particular activity, as exemplified by Protocol 13. There, the assimilation part-whole operations were sufficient for Jason to create a goal, defined there being a particular way of operating.

Third, there was Jason's implicit use of figurative patterns for "ten" in protocol 6. He organized his counting activity into two sequences of four abstract unit items (two

numerical patterns) and then a pair of abstract unit items. It was because he organized his counting activity in this way that I said the figurative pattern was implicit in his operating (cf. p. 34). The pattern for "ten" was active throughout his counting activity and provided a template he used to organize the activity. He was "inside" of the pattern, reprocessing its material into numerical subpatterns. This use of a "within stage" structure in the creation of novelties that led to vertical reorganization can be contrasted with the following use of the terms.

Fourth, I identified five stages of the counting scheme. In each stage, the products of operating in a preceding stage are made explicit in the current stage in the sense that they can be now taken as material of operating (they were used as material of reprocessing that led to a stage shift). As noted above, when children are in the stage of the initial number sequence, counting off from in subtraction is a method of subtracting for the children, but they have no awareness of why they subtract in that way or why it works. After constructing the tactilly nested number sequence, the children can make decisions concerning whether to count down to or to count off from when doing a subtraction problem and they can say why they operate as they do. But the "subtrahend" and the "difference" they create by subtracting are results that they will take as material of further operating when they construct the explicitly nested number sequence. At that stage, the children can use both of these parts and put them "back together" to form the minuend. This new power of operating permits the construction of the inversion between addition and subtraction, but the part-to-whole operations are yet to become material of operating.

Finally, I commented that double counting was implicit in Jason's counting acts and that the counting acts carried the force of double counting (p. 30). Moreover, I commented that the inclusion relation was implicit in the tactilly nested number sequence (cf. p. 39). This is a different use of the term than in the fourth case because there, for the operations involved in double counting to be implicit, the child would have to be able to double count -- "four in one, five in two, etc.". If I wanted to say that double counting was implicit for the child in the fourth sense of using the term, I would have to be able to make the inference that the child could double count and that he or she was aware of the results of double counting but could not use those results as material for further operating. The child would know how to double count and would be aware of double counting in that way. But they would not be aware of the structure of operating. This might seem to be confusing, but when I said that the counting acts carried the force of double counting, the comment was made from the perspective of that part of my mathematical knowledge that did not include Jason's mathematical knowledge. My analysis of "implicit" and "explicit" in the first four cases are from the point of view of the actor only--that part of my mathematical knowledge that does include the child's

mathematical knowledge

When the knowledge of tow people enter into consideration, the terms "explicit" and "implicit" take on new meaning. As observer, I say that double counting is **implicit** if the child puts up fingers to make records of counting backwards even though the child can't double count. The child's behavior **fits** my concept of double counting in that **I can see** coordinations the child makes, because putting up a finger when saying "sixteen" functions **as** a double count. The child's finger pattern for "eight" serves as an "automatic" counting device and makes the functional coordination possible. So, when the child is able function **as if** they were using operations that are not available to them, I as observer call those operations implicit. This may seem to be only semantics, but it is an important distinction because what an observer can "see" in the way a child currently functions may be a forerunner of novelties the child constructs in subsequent stages.

The distinction between what is implicit or explicit in one frame of reference vs. [illegible] reference is made from the point of view of the observer. Nevertheless, one [illegible] **within model** distinctions as well as **between model** distinctions. For example, from the perspective of **within model** distinctions, the initial number sequence is a reorganization of the figurative counting scheme and the tactily nested number sequence is a reorganization of the initial number sequence. From that perspective, we have to ask whether any number sequence is what Fischbein (1987) calls a primary intuition because novelties are introduced in the constructive process that have no natural roots, if "natural roots" is interpreted to mean roots in sensory experience. Double counting, for example, is introduced by the operating child through reprocessing re-presented segments of the initial number sequence using the unitizing operation. In other words, it is introduced by the child through operating on the products of past operating.

The initial number sequence is also constructed by the child through autoregulation of using the unitizing operation, so it too has no "natural roots". Because it is the first mathematical concept, my only conclusion is that none of mathematics can be categorized as a primary intuition if we speak from the point of view of the actor. From that point of view, if there is any part of mathematics that one would want to call intuitive, and that is certainly the case, it seems to me that it would be at least a secondary intuition. Mathematics **has no natural roots**, but is introduced as a novelty by the operating organism.

It is also important to note that while children are in a given learning stage of the counting scheme, they perform operations that engender the next stage. The products of

operating in a given stage prepare the way for the next stage, but a reorganization is required to achieve the next stage. So, even in this very elementary case, children reorganize their counting scheme as they achieve a new stage. The entire counting scheme is reorganized rather than pieces of it, which is what Fischbein takes as being necessary for a secondary intuition. The reorganization that leads to the explicitly nested number sequence is the fourth, so it might be called a fourth order intuition rather than a secondary intuition.

There is no necessity for anyone to construct mathematics, even the initial number sequence. But the observation that almost all human beings do construct elementary mathematical concepts and operations does make it reasonable to make a distinction between something like primary and secondary intuitions. I believe this distinction is based on two frames of reference--Fischbein's knowledge of mathematics (and physics) that does not include the mathematics of children and his knowledge of the mathematics of children. This is a fundamental and essential distinction that is especially important when considering what the mathematics of schooling might be.

Experiencing uncertainties

The question of how a child might become aware of an uncertainty in the results of operating is central to formulating a theory of cognitive constructions. Some progress can be made on the question in the particular case of Tyrone that I mentioned (cf. p. 27). Tyrone was yet to construct numerical patterns, but he had constructed figurative patterns as meanings of the number words through "five". His counting scheme was a figurative scheme and, like Jason, he was a counter of motor unit items. Given a collection of items to count where the items are hidden by two screens, one hiding seven and the other five, Tyrone tried to count the items beginning with "one". He characteristically pointed to the screen hiding the seven items synchronous with uttering the first seven number words and then continued on, pointing to the second screen synchronous with uttering "8, 9, ...". Tyrone's points of contact formed a row but there were no visible traces and hence no visual pattern that was obvious to the observer. Nevertheless, "five" had meaning for Tyrone in that he could visualize a domino five pattern upon hearing the word spoken and this pattern served in goal setting--count five more times. But, he could not stop counting at a specific counting act to fulfill his goal for the simple reason the figurative pattern was not used in keeping track of counting nor could it be used to recognize five counting acts. Nevertheless, it was an active goal. So, because there was no feedback from the results of counting to the goal that activated counting, the goal could not be satisfied by performing the activity. This creates an uncertainty in knowing when to stop counting. In fact, it lead to Tyrone reinitiating

his counting scheme and using his counting operation to reperceive re-presented results of counting in the creation of a feedback system.

Children's mathematics and education

Teaching experiments. Children's mathematics consists of the mathematical schemes of children that can function reliably and effectively and how those schemes might be modified as they are used. The methodology of the teaching experiment was developed for the explicit purpose of isolating those schemes and their modifications. A teaching experiment, even a two-year teaching experiment, usually is not taken to replace the overall educational experiences of children in mathematics. Rather, the goal is to study the mathematical operations of children and the teaching episodes constitute occasions for observation. The researchers do not assume that there is a causal relation between what transpires in the teaching episodes and what the child learns. Quite to the contrary, the teaching experiment is conducted in the context of the on going school program in mathematics and the changes that are observed in teaching episodes might well be a result of what transpired in the child's mathematics classroom. The teaching episodes are supplementary to the school mathematics program and so become a window on the mathematical operations children construct outside as well as within teaching episodes.

Because a teaching experiment consists of a sequence of teaching episodes, it is possible to observe modifications children make in their schemes. The goal is to learn the mathematics of children and this is why it is essential to be a participant in children's cognitive constructions. Piaget [1964] isolated experience as one crucial factor of intellectual development and this applies to researchers who aim at acquiring knowledge concerning children's conceptual structure and their evolution. My intention is to maximize my experience and reflection on that experience because the mathematical knowledge of children is always and inevitably seen in terms of my own frame of reference and can therefore be known only in terms of my own concepts and operations. There is no other way known to me that a viable theory of children's cognitive constructions in mathematics can be formulated.

A theory of cognitive constructions will always be relative to those who do the constructing. Researchers who build up a theory of children's mathematics can of course never discover what the children's 'real' knowledge is in an ontological sense. What we can and must try to do is to build a model that is compatible with our observations of children, and because we assume, as scientists do, that there is some regularity in our

experiential world, we can reach a point of relative closure after which we may feel justified to project the model into the future and to make predictions about further experiences with children. The "generality" of the models that are constructed rely on the theoretical constructs that are used in the conceptual analyses that comprise model building and on the powers of reasoning and insights of the model builders. But having something to explain is an essential part of building cognitive models—one must construct something like primary intuitions—and that is why theory and observation are mutually supporting.

The mathematics of schooling. There is always the question concerning the total educational experiences of children in mathematics and how the models constructed in a teaching experiment can contribute to these experiences. Personally, I can see no viable alternative to making children's mathematics the mathematics of schooling. To do less seems to me to continue to appeal to mathematics as we adults know it for the mathematics of schooling, a practice trivializes the mathematics that children do construct.

For example, children's number sequences in a given culture cannot be taken as being imbued with the all of their cultural meaning if for no other reason than the operations that they symbolize can be quite different among children to say nothing of the differences between children and adults. It is quite significant that the verbal number sequence was built up long before a mature system of written numerals became established.

> Our researches thus far have shown us that the laws that govern early numerals, ordering and grouping, do not in fact correspond to the rule of the sequence, which is stepwise gradation. Hence, the writing of numerals is not merely the representation of the number-word sequence (Menninger, 1958, p. 53).

Menninger's observation that Roman and Indo-European number sequences differ only in minor details, but that their systems of numeration differ radically, demonstrates the independence of verbal and written systems. It also provides a perspective on why there can be an unresolved tension between children's verbal number sequences and written numerals.

Whatever those numerical operations are that are available to children, they are initially symbolized by their verbal number sequences. Because of the way arithmetic is taught, these verbal number sequences often do not serve as a basis for children constructing the written system. The current "place value" approach to teaching number sequences used by most school mathematics programs serves as example. Although

logically impossible, this cardinal approach puts great demands on the children's counting operation. After making one ten, the children are to make one ten ... 1 one more (a progressive counting operation) and name that "eleven", and continue on until making two tens. They are then to take these two tens together as a unit to form a composite unit involving three ranks. They are to "see" twenty as a unit containing two units each of which contain ten units. To give meaning to "twenty" as "two tens", they have to unpack the containing unit into two tens and to further unpack the two tens and apprehend those singleton units as "twenty", because that was involved in establishing the two tens. This unit composition and decomposition is very demanding because it involves making a unit of units of units.

Children who have only constructed the initial number sequence must **make** a unit of units in activity and cannot take the resulting abstract composite unit as a given in further operating. So, we can see the large gap between the operations available to these children and what is demanded of them in the school program to work with "twenty". These children view "twenty" in terms of the **intensive** and **extensive meanings** that I have outlined **for units of one**. To organize these two meanings into a unit of units of units as I have described immediately above is not possible for these children in their immediate future. The situation is only a little better for children whose number words symbolize part-whole operations because they are also yet to construct a unit of units of units. They can find how many tens are in, say, one hundred by counting by tens, but they have to actually perform the activity to find that result. Prior to counting, they do not view 100 blocks, say, as being already partitioned into so many abstract composite units of ten. They can construct these partitioning operations, but that is another learning stage beyond the five I have already identified.

So, the results of teaching experiments can have fundamental importance for teaching mathematics in the elementary school if the mathematics of concern consists of children's mathematics. Given that this mathematics has been and will continue to be isolated in the interactive mathematical communication between adults and children (as well as among children) in learning environments, my hope is that it will be taken seriously by other mathematics educators as they strive to build educational programs in mathematics that are harmonious with children's ways and means of operating.

Notes

1 I would like to thank Dr. Iris Levin and Dr. Dina Tirosh for their insightful comments and criticisms on an earlier draft of this paper. Any inadequacies that remain are my responsibility.

2 Brenda was a participant in a two-year teaching experiment carried out to study the construction of numerical knowledge by six six-year-old children. The protocol is an excerpt from an interview held on October 21 of Brenda's first grade in school (Steffe & Cobb, 1988).

3 A re-presentation of a perceptual item is a recreation of the experience of the item in its absence.

4 A re-presentation of a sensory pattern is a recreation of the pattern without actual sensory material available.

5 A numerical finger pattern is created by applying the unitizing operation to a figurative finger patterns. Numerical finger patterns have a property of mobility and are not limited to specific finger patterns.

6 A counting word sequence is a number word sequence whose lexical items signify countable items.

7 Why a continuation of counting was activated rather than "1-2-3-4-5" can be understood if one thinks of Jason's figural collection of squares as being experientially separated. In this case, his intention could be to count all of the experientially separated squares starting with "one" and continuing until he counted all of them.

8 Toward the last part of the teaching experiment, Brenda did "count on" in some situations, but her behavior only indicated pseudo counting on—counting on beyond a pattern—because she did not independently produce counting on nor did she ever independently solve a "missing items" task.

9 The numerosity of a composite unit can be formed by instantiating a number sequence in an experiential content.

10 Jason's extensive and intensive meanings of number words are more compatible with ordinal number theory than they are cardinal number theory. However, I would not say that "sixteen" referred to an ordinal number if what is meant by that term is an object that is assigned to each well-ordered set in such a way that similar sets, and similar sets only, have the same object corresponding to them (Hausdorff, 1962, p. 65). Although there are many points of difference, Jason's **numerical concepts** as I have modeled them were not unlike ordinal numbers. An important feature of these numerical concepts was that he **focused on the constituent unit items** of the composite units he made rather than on the composite units as one thing.

11 A functional accommodation of a counting scheme is a modification of the scheme that occurs in the context of using the scheme. An engendering accommodation is a functional accommodation to self-initiated, involves using conceptual elements external as well as internal to the scheme, leads to further accommodations, and involves or leads to a structural reorganization of the scheme.

12 A metamorphic accommodation of a counting scheme is a modification of the scheme that occurs independently but not to any particular applications of the scheme.

13 I use parentheses to indicate that Jason took the results of counting as one thing, which is to say that he took his finger pattern as one thing.

14 It is understandable why Jason said "16-15-14-13" to find how many pokerchips remained under the cloth in protocol 9 because he had just counted to "sixteen" to place sixteen poker chips in the cup. Experientially, he was at the "end" of the verbal number sequence which now symbolized counting the chips. Any chip could correspond to "sixteen" because that word symbolized a counting act. He uttered the four counting words backward to find where he was in counting before he counted the four visible chips.

15 A procedural accommodation is a functional accommodation that involves a modification of the activity of counting or a novel way to view the results of counting. A procedural accommodation might be an engendering accommodation.

16 Sicenstra solved "98 - 37 = _____" by ticking off fingers synchronous with uttering "47-46-45-44-43-42-41-40-39-38-37" and then said "eleven", recognizing the resulting finger pattern. Immediately afterwards, she solved "53 - 12 = _____" by ticking off fingers synchronous with uttering "52-51-...-42". She passed over her thumb without saying "forty-two", a simple error in coordination.

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